Effect of fluid flow induced by a rotating disk on the freezing of fluid

Joo Sik Yoo
Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology, Chongryang, Seoul, Korea

This article investigates the effect of the fluid flow induced by an infinite rotating disk on the freezing of the fluid and the effect of the freezing on the transient heat transfer from the fluid to the disk. The Von Kármán similarity solution is used for the velocity of the fluid. The transient behavior of the temperature distribution in both solid and liquid phases and the freezing rate are determined.

Keywords: phase change; freezing; growth of solid

1. Introduction

Heat transfer problems in the freezing (or melting) process have attracted considerable attention in view of both their theoretical interest and their practical applications, e.g., crystal growth, casting, and welding. Since the work by Stefan (Carslaw and Jaeger[1]), a great number of analytical and numerical studies on heat transfer problems with phase change have been performed to determine the temperature distributions in solid and stationary liquid (Muehlbauer and Sunderland[2]). Many investigators have also considered the phase-change problem with forced convection[3–5] or natural convection[6–8] in the melt. In all the works with forced convection, as far as the present author is aware, the rate of convective heat transfer from the liquid side of the boundary of phase change was assumed to be known (constant), and the temperature distribution in the solid and the location of the solid–liquid interface were determined.[3–5] It is evident, however, that the freezing process can be affected by the transient development of temperature distribution in the liquid and vice versa.

In this study, the freezing of an incompressible fluid in motion induced by an infinite rotating plane disk is considered. The disk is rotating slowly in its own plane with constant angular velocity Ω. The flow is laminar; accordingly, the velocity field is described by the Von-Kármán similarity equations (Zandbergen and Dijkstra[9]). Initially ($t=0$), the fluid and the disk are kept at a uniform temperature (T_h) higher than the freezing temperature of the fluid (T_f). For $t>0$, the temperature of the disk is lowered to T_c ($T_c<T_f$) and maintained constant. The transient behavior of the temperature distribution in both solid and liquid phases and the freezing rate are determined. Primary attention is given to the growth of the solid.

Although the present problem is very fundamental in the phase-change problem with forced convection, there has been neither theoretical nor experimental study about it, as far as the present author is aware. The present problem has two notable aspects of fundamental research: (1) if there is no fluid flow, then it reduces to the well-known Neumann problem (Carslaw and Jaeger[1]), and (2) if no phase change is present, then it becomes the problem of transient heat transfer in the rotating-disk–revolving-fluid system (Olander[10]). In the practical aspect, the present work is related to the Czochralski growth,[11–15] where the crystal is rotated slowly to obtain homogeneous crystal from melt.

In Section 2, the governing equations and the boundary conditions are presented. We can obtain similarity Equations 7–11. Some analytic expressions and numerical methods used are described in Section 3. Section 4.1 displays the results for the effect of the fluid flow on the growth of solid. The results for the heat transfer problem in the rotating-disk–revolving-fluid system with phase change are presented in Section 4.2.

2. Formulation of the problem

The physical system considered is shown in Figure 1. The thermophysical properties of solid and liquid phases are assumed constant, and the density change of the material upon freezing is neglected so that there is no fluid flow induced by the volumetric change in the phase-change process.[1–8] The thickness of the solidified layer $X(t)$ is assumed not to vary spatially, since the temperature distribution in the liquid phase can be considered one-dimensional (1-D).[10–16] From the exact solution of Neumann,[1] we know that the solid grows very slowly. On the other hand, even the fluid flow induced by the impulsive rotating of the disk initially at rest approaches its asymptotic steady state after only about 2 radians of the disk's motion (Benton[17]). Thus, we assume that the fluid flow is steady and

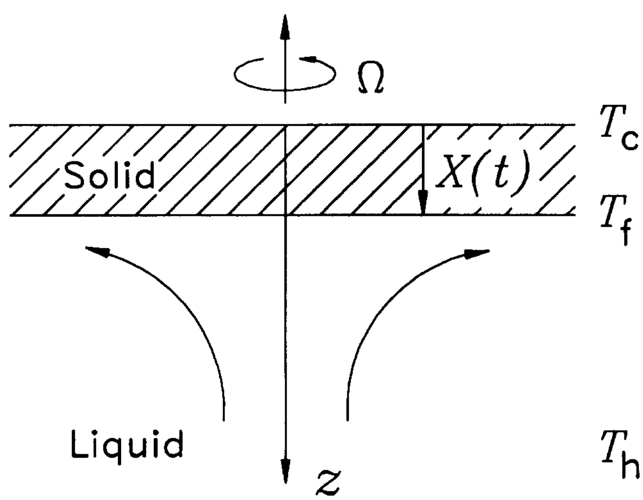

Figure 1 Problem configuration

Address reprint requests to Dr. Yoo at the Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology, PO Box 150, Chongryang, Seoul, Korea.

Received 31 August 1990; accepted 18 January 1991

is not influenced by the movement of the solid–liquid interface. Under these assumptions, the flow field is described by the Von-Kármán similarity solution.

The temperature distribution in the liquid phase (T_l) is governed by

$$\frac{\partial T_l}{\partial t}+w(z-X(t))\frac{\partial T_l}{\partial z}=\alpha_l\frac{\partial^2 T_l}{\partial z^2} \tag{1}$$

where $w(z)$ denotes the velocity component in the direction normal to the disk. Numerical values of $w(z)$ are given by Benton.[17]

$$w(z)=(\nu\Omega)^{1/2}H(\xi);\qquad \xi=(\Omega/\nu)^{1/2}z \tag{2}$$

$$H(0)=H'(0)=0,\qquad H''(0)=-1.0205$$

$$H(\xi)\simeq c\{-1+2.364\exp(-c\xi)-1.880\exp(-2c\xi)+\cdots\},\quad \text{as } \xi\to\infty$$

$$c=0.88447$$

In the solid phase, the temperature distribution (T_s) is described by

$$\frac{\partial T_s}{\partial t}=\alpha_s\frac{\partial^2 T_s}{\partial z^2} \tag{3}$$

At the solid–liquid interface $z=X(t)$, where the change of state occurs, the energy balance is maintained:

$$K_s\frac{\partial T_s}{\partial z}-K_l\frac{\partial T_l}{\partial z}=\rho L\frac{dX}{dt} \tag{4}$$

Additional boundary conditions for T_s and T_l are

$$T_s=T_c \text{ at } z=0,\ T_s=T_l=T_f \text{ at } z=X(t),\ T_l\to T_h \text{ as } z\to\infty \tag{5}$$

Let us introduce following dimensionless variables:

$$\tau=\Omega t,\qquad \eta=z/X(t),$$

$$\theta_s=(T_s-T_c)/(T_f-T_c),\qquad \theta_l=(T_l-T_h)/(T_f-T_h), \tag{6}$$

$$K_r=K_s/K_l,\qquad \alpha_r=\alpha_s/\alpha_l,\qquad \mathrm{Pr}=\nu/\alpha_l,$$

$$\theta_r=(T_h-T_f)/(T_f-T_c),\qquad \mathrm{Ste}=c_s(T_f-T_c)/L$$

Equations 1 and 3 are rewritten as

$$h_s^2\frac{\partial\theta_s}{\partial\tau}-\frac{\eta}{2}\frac{dh_s^2}{d\tau}\frac{\partial\theta_s}{\partial\eta}=\alpha_r\frac{\partial^2\theta_s}{\partial\eta^2} \tag{7}$$

$$h_s^2\frac{\partial\theta_l}{\partial\tau}-\frac{\eta}{2}\frac{dh_s^2}{d\tau}\frac{\partial\theta_l}{\partial\eta}+\mathrm{Pr}^{1/2}\cdot h_s\cdot H\{\mathrm{Pr}^{-1/2}(\eta-1)h_s\}\frac{\partial\theta_l}{\partial\eta}=\frac{\partial^2\theta_l}{\partial\eta^2} \tag{8}$$

where

$$h_s(\tau)=(\Omega/\alpha_l)^{1/2}X(t) \tag{9}$$

denotes the dimensionless thickness of the frozen layer. It is assumed that $h_s(0)=0$.

Boundary conditions (4) and (5) are transformed:

$$\frac{\partial\theta_s}{\partial\eta}+\frac{\theta_r}{K_r}\frac{\partial\theta_l}{\partial\eta}=\frac{1}{2\alpha_r\mathrm{Ste}}\frac{dh_s^2}{d\tau}\quad \text{at } \eta=1 \tag{10}$$

$$\theta_s=0 \text{ at } \eta=0,\quad \theta_s=\theta_l=1 \text{ at } \eta=1,\quad \theta_l\to 0 \text{ as } \eta\to\infty. \tag{11}$$

Note that θ_r/K_r is a parameter in the governing Equations 7–11, because the heat transfer rate is determined by Fourier's law of heat conduction.

3. Analysis

At the initial stage of freezing ($\tau\ll 1$), $\theta_{s,l}(\tau,\eta)$ and $h_s(\tau)$ can be expanded as follows:

$$\theta_{s,l}(\tau,\eta)=\theta^0_{s,l}(\eta)+\theta^1_{s,l}(\eta)\tau^{3/2}+\cdots \tag{12}$$

$$h_s^2(\tau)=b_0\tau+b_1\tau^{5/2}+\cdots \tag{13}$$

For small τ, the temperature front at which $\theta_l=0$ ($=\theta_\infty$) is near the interface, and thus the velocity field in the liquid is approximated:

$$H(\xi)=H''(0)\xi^2/2 \tag{14}$$

Substitution of Equations 12–14 into Equations 7–11 yields

Notation

b_0, b_1	Constants in Equation (13)
C_1, C_2, C_3, C_4	Constants in Equations (29) and (30)
c	Magnitude of the velocity of fluid at $z\to\infty$
c_s	Specific heat of solid
erf	Error function
erfc	Complementary error function
$H(\xi)$	Dimensionless axial velocity component of fluid
$h_s(\tau)$	Dimensionless thickness of frozen layer, $(\Omega/\alpha_l)^{1/2}X(t)$
h_{eq}	Steady-state value of $h_s(\tau)$
K	Thermal conductivity
K_r	Ratio of thermal conductivity, K_s/K_l
L	Latent heat
Log(x)	$\mathrm{Log}_{10}(x)$
Nu	Nusselt number, $\partial\theta_l/\partial\xi$ at the interface
Nu_{ss}	Nusselt number in steady state
Pr	Prandtl number, ν/α_l
Ste	Stefan number, $c_s(T_f-T_c)/L$
T	Temperature
T_c, T_f, T_h	Cold, freezing, and hot temperature, respectively
$w(z)$	Axial velocity component of fluid
$X(t)$	Thickness of frozen layer
X_{eq}	Steady-state value of $X(t)$
z	Axial coordinate

Greek symbols

α	Thermal diffusivity
α_r	Ratio of thermal diffusivity, α_s/α_l
ξ	Dimensionless axial coordinate, $(\Omega/\nu)^{1/2}z$
η	Dimensionless coordinate, $z/X(t)$
θ_l	Dimensionless temperature in liquid, $(T_l-T_h)/(T_f-T_h)$
θ_s	Dimensionless temperature in solid, $(T_s-T_c)/(T_f-T_c)$
θ_r	Ratio of temperature, $(T_h-T_f)/(T_f-T_c)$
ν	Kinematic viscosity of liquid
ρ	Density, $\rho_s=\rho_l=\rho$
σ	Growth parameter in Neumann problem
τ	Dimensionless time, Ωt
Ω	Angular velocity of rotating disk

Subscripts

l	Liquid
s	Solid
∞	Infinity

$O(\tau^0)$ equations

$$\frac{d^2\theta_s^0}{d\eta^2}+\frac{b_0}{2\alpha_r}\eta\frac{d\theta_s^0}{d\eta}=0 \quad \text{at } 0<\eta<1 \tag{15}$$

$$\frac{d^2\theta_l^0}{d\eta^2}+\frac{b_0}{2}\eta\frac{d\theta_l^0}{d\eta}=0 \quad \text{at } \eta>1 \tag{16}$$

$$\frac{d\theta_s^0}{d\eta}+\frac{\theta_r}{K_r}\frac{d\theta_l^0}{d\eta}=\frac{b_0}{2\alpha_r\mathrm{Ste}} \quad \text{at } \eta=1 \tag{17}$$

$$\theta_s^0(0)=0, \quad \theta_s^0(1)=\theta_l^0(1)=1, \quad \theta_l^0(\infty)=0 \tag{18}$$

and $O(\tau^{3/2})$ equations

$$\frac{d^2\theta_s^1}{d\eta^2}+\frac{b_0}{2\alpha_r}\eta\frac{d\theta_s^1}{d\eta}-\frac{3b_0}{2\alpha_r}\theta_s^1=-\frac{5b_1}{4\alpha_r}\eta\frac{d\theta_s^0}{d\eta} \quad \text{at } 0<\eta<1 \tag{19}$$

$$\frac{d^2\theta_l^1}{d\eta^2}+\frac{b_0}{2}\eta\frac{d\theta_l^1}{d\eta}-\frac{3b_0}{2}\theta_l^1=\left[\frac{H''(0)}{2}\mathrm{Pr}^{-1/2}b_0^{3/2}(\eta-1)^2-\frac{5b_1}{4}\eta\right]\frac{d\theta_l^0}{d\eta} \quad \text{at } \eta>1 \tag{20}$$

$$\frac{d\theta_s^1}{d\eta}+\frac{\theta_r}{K_r}\frac{d\theta_l^1}{d\eta}=\frac{5b_1}{4\alpha_r\mathrm{Ste}} \quad \text{at } \eta=1 \tag{21}$$

$$\theta_s^1(0)=\theta_s^1(1)=\theta_l^1(1)=\theta_l^1(\infty)=0 \tag{22}$$

The solution of Equations 15–18 is the well-known Neumann solution.[1]

$$\theta_s^0(\eta)=\mathrm{erf}(\sigma\eta)/\mathrm{erf}(\sigma) \tag{23}$$

$$\theta_l^0(\eta)=\mathrm{erfc}(\sigma\alpha_r^{1/2}\eta)/\mathrm{erfc}(\sigma\alpha_r^{1/2}) \tag{24}$$

$$\frac{\exp(-\sigma^2)}{\mathrm{erf}(\sigma)}-\frac{\theta_r\alpha_r^{1/2}\exp(-\sigma^2\alpha_r)}{K_r\,\mathrm{erfc}(\sigma\alpha_r^{1/2})}=\frac{\pi^{1/2}\sigma}{\mathrm{Ste}} \tag{25}$$

$$b_0=4\sigma^2\alpha_r \tag{26}$$

The solution of Equations 19–22 is found with a homogeneous solution of the form

$$\theta_s^{1h}(\eta)=\eta^3+\frac{3}{2\sigma^2}\eta=u_s(\eta) \tag{27}$$

$$\theta_l^{1h}(\eta)=\eta^3+\frac{3}{2\sigma^2\alpha_r}\eta=u_l(\eta) \tag{28}$$

This gives

$$\theta_s^1(\eta)=-u_s(\eta)\int_\eta^1\frac{b_1C_1\left(\dfrac{\eta^5}{5}+\dfrac{\eta^3}{2\sigma^2}\right)}{\exp(\sigma^2\eta^2)u_s^2(\eta)}d\eta \tag{29}$$

$$\theta_l^1(\eta)=u_l(\eta)\int_1^\eta\frac{C_3f(\eta)+b_1C_2\left(\dfrac{\eta^5}{5}+\dfrac{\eta^3}{2\sigma^2\alpha_r}\right)+C_4}{\exp(\sigma^2\alpha_r\eta^2)u_l^2(\eta)}d\eta \tag{30}$$

where

$$C_1=-(5\sigma)/\{2\pi^{1/2}\alpha_r\,\mathrm{erf}(\sigma)\}$$

$$C_2=(5\sigma\alpha_r^{1/2})/\{2\pi^{1/2}\,\mathrm{erfc}(\sigma\alpha_r^{1/2})\}$$

$$C_3=-\{8H''(0)\sigma^4\alpha_r^2\}/\{\pi^{1/2}\mathrm{Pr}^{1/2}\,\mathrm{erfc}(\sigma\alpha_r^{1/2})\}$$

$$f(\eta)=\frac{\eta^6}{6}-\frac{2}{5}\eta^5+\frac{1}{4}\left(1+\frac{3}{2\sigma^2\alpha_r}\right)\eta^4-\frac{1}{\sigma^2\alpha_r}\eta^3+\frac{3}{4\sigma^2\alpha_r}\eta^2$$

The remaining unknowns C_4 and b_1 are determined by boundary condition (21) and $\theta_l^1(\infty)=0$. It is readily seen that

$$b_1=F(\alpha_r,\theta_r/K_r,\mathrm{Ste})\times(\mathrm{Pr})^{-1/2} \tag{31}$$

The processes are straightforward, and the detailed equations are omitted for brevity.

For large time, the solution of Equations 1–5 is obtained numerically. There are several numerical methods for the moving boundary problems.[18] Two completely different methods, namely, the enthalpy method (Voller and Cross[19]) and the method using body-fitted coordinates (Sparrow et al.[20]), were tested. The accuracy of the numerical schemes was checked with the exact solution of Neumann and the steady-state solution of Equations 1–5. The enthalpy method yielded accurate solutions for small time, but not for large time, as Bell[21] pointed out. On the other hand, the method using body-fitted coordinates yielded accurate solutions up to very long time and was very efficient, since it allowed large time steps. In solving the Neumann problem, agreement with the exact solution to within 0.1% was attained for the thickness of the frozen layer $X(t)$; when there was fluid flow, the solution approached the exact steady-state solution as time went on. The governing Equations 7–11 that were obtained by a dimensionless coordinate $\eta=z/X(t)$ were solved with this method. The moving boundary was fixed at $\eta=1$ for all times. And in the numerical procedure, the dimensionless time $\tau'=(\mathrm{Ste}/\mathrm{Pr})\tau$ made it easy to choose time steps. One hundred grid points were developed uniformly throughout the solid region, and 500 grid points were developed nonuniformly throughout the liquid region according to the relation $\eta_i=1+(\eta_\infty-1)\{(i-1)/499\}^{1.7}$. An implicit finite-difference scheme was used for the energy Equations 7 and 8, and explicit representation was used for the interfacial energy balance (Equation 10). The resulting difference equations for the temperature distribution θ_s and θ_l were solved noniteratively at each time step by using the tridiagonal matrix algorithm. And h_s and $dh_s/d\tau$ were determined from Equation 10 with the method used by Sparrow et al.[20]

4. Results and discussion

4.1. Effect of fluid flow on the growth of solids

At the initial stage of freezing, the thickness of the frozen layer is approximated as $h_s^2(\tau)=b_0\tau+b_1\tau^{5/2}+\cdots$, where the first term represents the pure conduction solution of Neumann[1] and the second represents the effect of the fluid flow. Calculation shows that b_1 has negative values in all cases considered. The magnitude of (b_1/b_0) is increased as Ste or θ_r/K_r becomes large. This reveals that as initial liquid temperature (T_h), conductivity of the liquid, or Stefan number becomes larger, the growth of the solid is more strongly inhibited by the fluid flow. And b_1 is proportional to $\mathrm{Pr}^{-1/2}$ from Equation 31. Thus the effect of the fluid flow on the growth of the solid is decreased as the viscosity of the fluid increases. This is because the thickness of the velocity boundary layer is proportional to $\nu^{1/2}$.

As time goes on, $h_s(\tau)$ increases monotonously and approaches the steady-state value h_{eq} (Figure 2a). The growth rate of the solid is increased as α_r or Ste becomes large, but is decreased as θ_r/K_r or Pr becomes large. The effect of the solid rotation (Ω) on the growth of the solid can be seen with the dimensional variables $t=(\tau/\Omega)$ and $X(t)=(\alpha_l/\Omega)^{1/2}h_s(\tau)$. To see this effect, an example problem was devised with the thermophysical properties of the silicon used in the Czochralski growth. Figure 2a represents the dimensionless solid length $h_s(\tau)$ as a function of dimensionless time τ, and Figure 2b was obtained from this result. The curve of $\Omega=0$ represents the growth of the solid when there is no fluid flow, and other curves clearly show the effect of the solid rotation (Ω) on the growth of the solid. That is, the fluid flow induced by the solid rotation (Ω) strongly inhibits the freezing of the fluid. This is also demonstrated with the approximate solution for small time. Equation 13 yields

$$X(t)=2\sigma(\alpha_s t)^{1/2}\{1-|b_1/2b_0|(\Omega t)^{3/2}\} \tag{32}$$

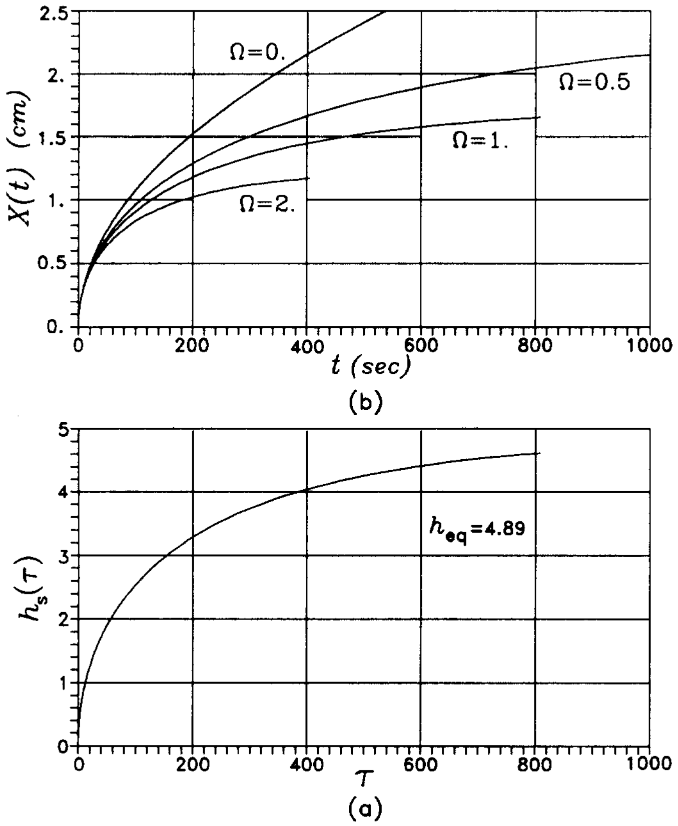

Figure 2 Growth of solid as a function of time. The material for this example is silicon, whose approximate thermophysical properties near the freezing temperature $T_f = 1685$(K) are $c_s = 1$ (J/g·K), $K_s = 0.22$(W/cm·K), $K_l = 0.32$(W/cm·K), $L = 1800$(J/g), $\alpha_s = 0.088$(cm²/s), $\alpha_l = 0.128$(cm²/s), and Pr = 0.02. Cold and hot temperatures are taken as $T_c = 1500$(K) and $T_h = 1900$(K), respectively. (a) Dimensionless solid length ($h_s(\tau)$) as a function of dimensionless time (τ); (b) dimensional solid length ($X(t)$) as a function of dimensional time (t) for several angular velocities. Unit of Ω is rad/s

In the Neumann problem,[1] there is no steady state, and the solid grows continuously with time according to the relation $X(t) = 2\sigma(\alpha_s t)^{1/2}$. In the present problem, however, the fluid flow restricts the propagation of the thermal boundary in the liquid; accordingly, the system approaches a final equilibrium state (steady state) as time goes on. Equations 1–5 with $\partial/\partial t = 0$ give the steady state. The steady-state thickness of the frozen layer X_{eq} is found from the energy balance at the interface (Equation 4). This gives

$$X_{eq} = -(K_r/\theta_r)\cdot(\nu/\Omega)^{1/2} \bigg/ \left(\frac{d\theta_l}{d\xi}\bigg|_{\text{interface}}\right) \tag{33}$$

Thus X_{eq} is proportional to $\Omega^{-1/2}$. For the limiting case of $\Omega \to 0$ or $\theta_r \to 0$, the solution becomes identical to that of Neumann ($X_{eq} \to \infty$).

Many investigators[3-5] considered the freezing of fluid in forced flow with a given rate of convective heat transfer from the liquid side of the solid–liquid interface. In Figure 3, the time required to reach $h_s/h_{eq} = 0.95$ and that required to reach $Nu/Nu_{ss} = 1.05$ ($Nu = \partial\theta_l/\partial\xi$ at the interface) are shown as functions of Ste and θ_r/K_r. These results show that for small Ste or θ_r/K_r, the Nusselt number reaches its quasi-steady-state value much faster than $h_s(\tau)$, and thus we can consider it to be constant, $Nu(t) \simeq Nu_{ss}$, throughout the freezing process. However, for large Ste or θ_r/K_r, the transient behavior of Nu(t) should be considered.

4.2. Effect of phase change on the heat transfer problem in the rotating-disk–revolving-fluid system

The transient heat transfer behavior in the rotating-disk–revolving-fluid system without phase change was considered by a few authors.[10,16] However, if the temperature imposed on the disk is lower than the freezing temperature of the fluid, the fluid close to the disk would freeze. When there are such phase changes, the time evolution of Nusselt numbers for Pr = 0.01, 0.1, 1, 10, and 100 is shown in Figure 4. The transient behavior is similar to that observed when there is no phase change, but it takes a little longer time to reach the same value of Nu/Nu_{ss} than the case with no phase change (Olander[10]). In Figure 5, the time required to reach $Nu/Nu_{ss} = 1.05$ is given for some Stefan numbers as a function of Prandtl number. Curve (a) represents the case without phase change (Olander[10]). When phase change is present, it takes a longer time to reach the quasi-steady state ($Nu/Nu_{ss} = 1.05$) than when there is no phase change. This is because the freezing of the fluid hinders the propagation of the thermal boundary from the interface to the

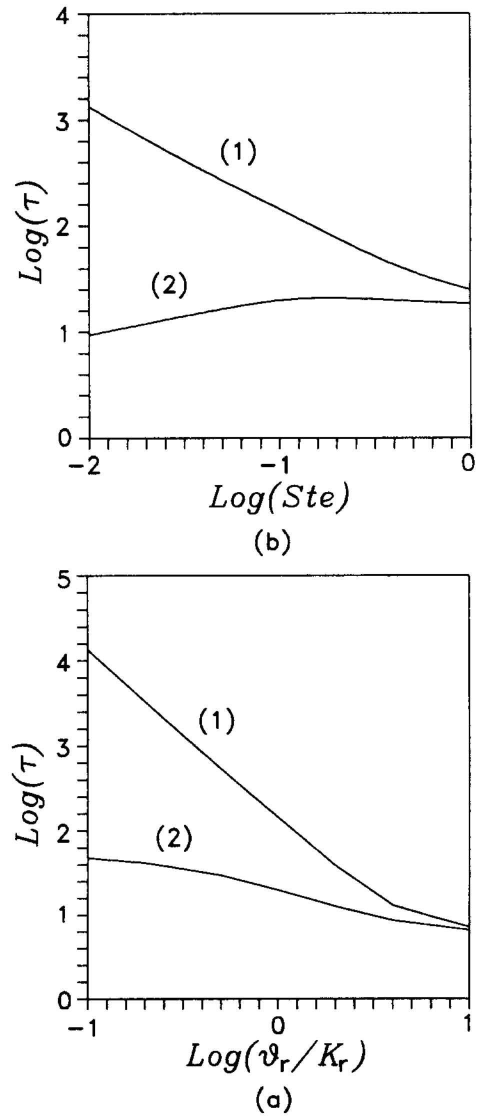

Figure 3 Time (τ) required to reach $h_s(\tau)/h_{eq} = 0.95$ (1) and that to reach $Nu(\tau)/Nu_{ss} = 1.05$ (2). (a) (θ_r/K_r)—dependency, with α_r = Pr = 1, Ste = 0.1; (b) (Ste)—dependency, with $\alpha_r = \theta_r/K_r$ = Pr = 1

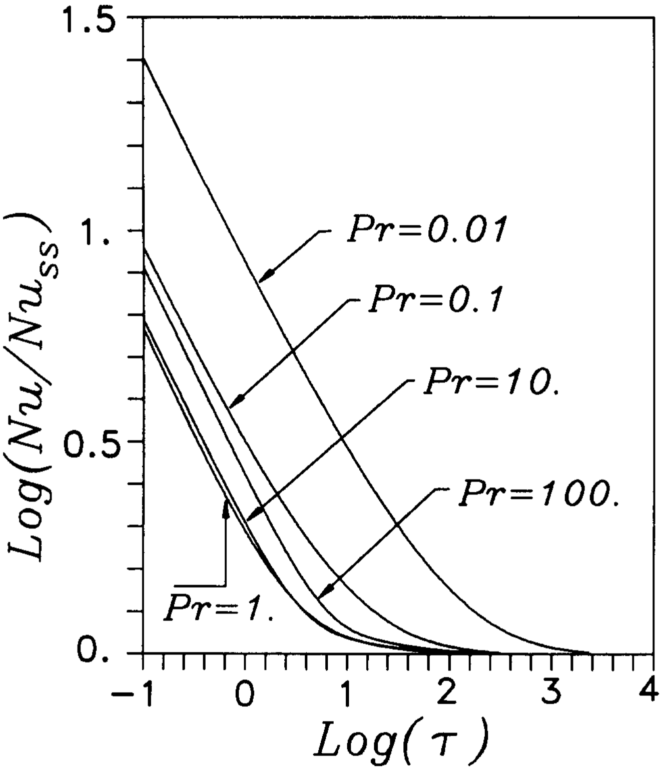

Figure 4 Transient heat transfer ($Nu(\tau)/Nu_{ss}$) from the liquid side of solid–liquid interface for Pr=0.01, 0.1, 1, 10, and 100 with $\alpha_r=\theta_r/K_r=1$, Ste=0.1

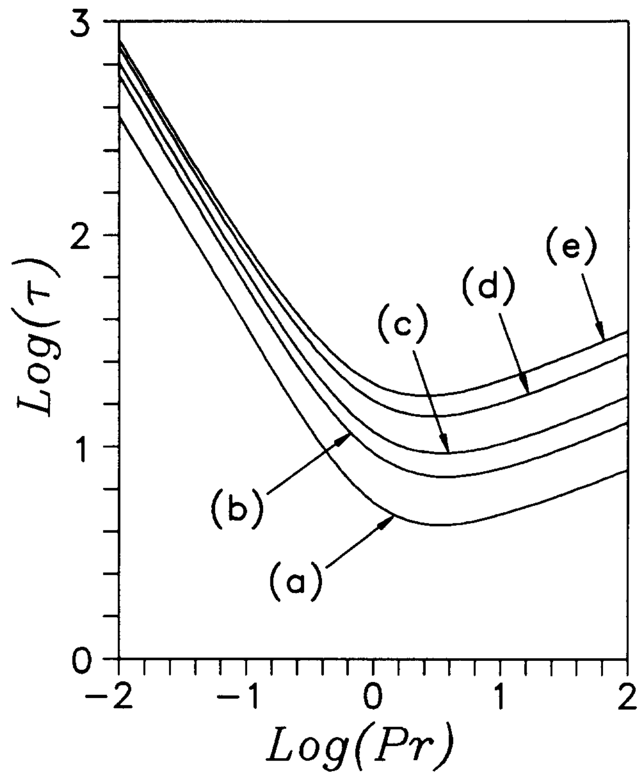

Figure 5 Time (τ) required to reach $Nu(\tau)/Nu_{ss}=1.05$ as a function of Pr for several Stefan numbers with $\alpha_r=\theta_r/K_r=1$. (a) The case with no phase change; (b) Ste=0.01; (c) Ste=0.02; (d) Ste=0.05; (e) Ste=0.1

liquid region. The time required to reach the quasi-steady state becomes longer as the Stefan number becomes larger.

5. Concluding remark

The rotation (Ω) of a solid strongly inhibits the freezing of fluid, since the fluid flow induced by the solid rotation transports hot fluid toward the solid–liquid interface where phase change occurs. As Ste or θ_r/K_r becomes larger, the freezing of the fluid is more strongly inhibited by the fluid flow. For small Ste or θ_r/K_r, the rate of heat transfer from the liquid side of the solid–liquid interface can be considered constant ($Nu=Nu_{ss}$) throughout the freezing process, but for large Ste or θ_r/K_r, the transient behavior of it should be considered. As Ste becomes larger, the time required to reach quasi-steady state ($Nu/Nu_{ss}=1.05$) becomes longer.

Acknowledgment

I would like to thank Professor Moon-Uhn Kim in the Department of Applied Mathematics for helpful discussions on this problem.

References

1 Carslaw, H. S. and Jaeger, J. C. *Conduction of Heat in Solids*, 2nd Ed., Clarendon Press, Oxford, 1959

2 Muehlbauer, J. C. and Sunderland, J. E. Heat conduction with freezing or melting. *Appl. Mech. Rev.*, 1965, **18**, 951–959

3 Beaubouef, R. T. and Chapman, A. J. Freezing of fluids in forced flow. *Int. J. Heat Mass Transfer*, 1967, **10**, 1581–1587

4 Savino, J. M. and Siegel, R. An analytical solution for solidification of a moving warm liquid onto an isothermal cold wall. *Int. J. Heat Mass Transfer*, 1969, **12**, 803–809

5 Epstein, M. The growth and decay of a frozen layer in forced flow. *Int. J. Heat Mass Transfer*, 1976, **19**, 1281–1288

6 Sparrow, E. M., Patankar, S. V., and Ramadhyani, S. Analysis of melting in the presence of natural convection in the melt region. *J. Heat Transfer, Trans. ASME*, Series C, 1977, **99**, 520–526

7 Ho, C. J. and Chen, S. Numerical simulation of melting of ice around a horizontal cylinder. *Int. J. Heat Mass Transfer*, 1986, **29**, 1359–1369

8 Benard, C., Gobin, D., and Zanoli, A. Moving boundary problem: heat conduction in the solid phase of a phase-change material during melting driven by natural convection in the liquid. *Int. J. Heat Mass Transfer*, 1986, **29**, 1669–1681

9 Zandbergen, P. J. and Dijkstra, D. Von Kármán swirling flows. *Annu. Rev. Fluid Mech.*, 1987, **19**, 465–491

10 Olander, D. R. Unsteady-state heat and mass transfer in the rotating-disk–revolving-fluid system. *Int. J. Heat Mass Transfer*, 1962, **5**, 825–836

11 Langlois, W. E. Convection in Czochralski growth melts. *PhysicoChem. Hydrodynamics*, 1981, **2**, 245–261

12 Burton, J. A., Prim, R. C. and Slichter, W. P. The distribution of solute in crystals growth from melt. Part I. Theoretical. *J. Chem. Phys.*, 1953, **21**, 1987–1991

13 Wilson, L. O. On interpreting a quantity in the Burton, Prim and Slichter equation as a diffusion boundary layer thickness. *J. Crystal Growth*, 1978, **44**, 247–250

14 Favier, J. J. and Wilson, L. O. A test of the boundary layer model in unsteady Czochralski growth. *J. Crystal Growth*, 1982, **58**, 103–110

15 Kobayashi, S. Effects of an external magnetic field on solute distribution in Czochralski growth crystals—a theoretical analysis. *J. Crystal Growth*, 1986, **75**, 301–308

16 Homsy, G. M. and Hudson, J. L. Unsteady heat transfer from

a rotating disk. *J. Heat Transfer, Trans. ASME*, Series C, 1969, **91**, 162–163

17 Benton, E. R. On the flow due to a rotating disk. *J. Fluid Mech.*, 1966, **24**, 781–800

18 Crank, J. How to deal with moving boundaries in thermal problems. In *Numerical Method in Heat Transfer*, R. W. Lewis, K. Morgan, and O. C. Zienkiewicz, eds. John Wiley and Sons Ltd, New York, 1981

19 Voller, V. and Cross, M. Accurate solutions of moving boundary problems using the enthalpy method. *Int. J. Heat Mass Transfer*, 1981, **24**, 545–556

20 Sparrow, E. M., Ramadhyani, S., and Patankar, S. V. Effect of subcooling on cylindrical melting. *J. Heat Transfer, Trans. ASME*, Series C, 1978, **100**, 395–402

21 Bell, G. E. On the performance of the enthalpy method. *Int. J. Heat Mass Transfer*, 1982, **25**, 587–589